Gov 51: Conditional Probability and Independence

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Conditional probability

- If we know that B has occurred, what is the probability of A?
  - Conditioning our analysis on B having occurred.
- Examples:
  - Probability of two states going to war *if* they are both democracies?
  - Probability of a judge issuing a pro-choice ruling *if* they have daughters?
  - Probability of a coup in a country *if* it has a presidential system?
- Conditional probability extremely useful for data analysis.

Conditional Probability definition

- Definition: If $\mathbb{P}(B) > 0$ then the **conditional probability** of A given B is

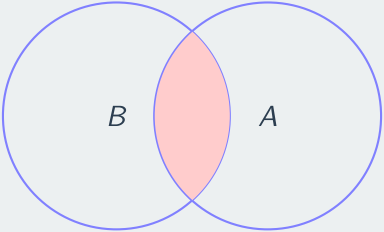
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

- How often A and B occur divided by how often B occurs.
- **WARNING!** $\mathbb{P}(A \mid B)$ does **not**, in general, equal $\mathbb{P}(B \mid A)$.
  - $\mathbb{P}(\text{smart} \mid \text{in gov 51})$ is high
  - $\mathbb{P}(\text{in gov 51} \mid \text{smart})$ is low.
  - There are many many smart people who are not in this class!
- If all outcomes equally likely:

$$\mathbb{P}(A \mid B) = \frac{\text{number of outcomes in both } A \text{ and } B}{\text{number of outcomes in just } B}$$

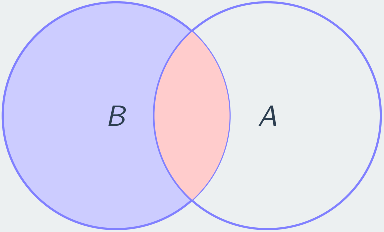
Conditional probability

$P(A \text{ and } B)$



Conditional probability

$$P(A \mid B)$$



US Senate example

	Democrats	Republicans	Independents	Total
Men	28	44	2	74
Women	17	9	0	26
Total	45	53	2	100

- Choose one senator at random from this population
- What is the probability of choosing a woman?
  - $\mathbb{P}(\text{Woman}) = \frac{26}{100} = 0.26$
- What is the probability of choosing a Republican who is a woman?
  - $\mathbb{P}(\text{Woman and Republican}) = \frac{9}{100} = 0.09$
- What is the probability that a randomly selected Republican is a woman:
  - $\mathbb{P}(\text{Woman} \mid \text{Rep.}) = \frac{\mathbb{P}(\text{Woman and Rep.})}{\mathbb{P}(\text{Rep.})} = \frac{9/100}{53/100} = \frac{9}{53} \approx 0.17$

Conditional probability rules

- Multiplication rule:

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A \mid B)\mathbb{P}(B) = \mathbb{P}(B \mid A)\mathbb{P}(A)$$

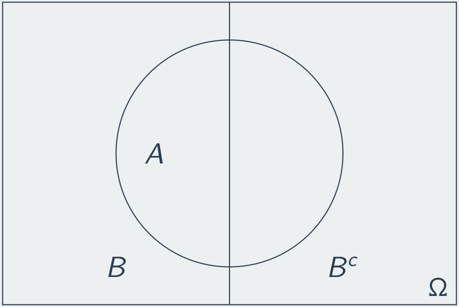
Multiplication rule, example

	Democrats	Republicans	Independents	Total
Men	28	44	2	74
Women	17	9	0	26
Total	45	53	2	100

- Draw the names of two senators from a hat.
- What's the probability that we draw two women?
  - Let W_1 and W_2 be the events that 1st and 2nd draws are women.
  - We could make a list of all possible pairs to draw and count them...
  - Or we could just use the multiplication rule:

$$\mathbb{P}(W_1 \text{ and } W_2) = \mathbb{P}(W_1)\mathbb{P}(W_2 \mid W_1)$$

Law of Total Probability



- Conditional probability lets us restate the law of total probability.
- **Law of total probability**:

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A \text{ and } B) + \mathbb{P}(A \text{ and not } B)\\ &= \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid \text{not } B)\mathbb{P}(\text{not } B)\end{aligned}$$

Independence

- Two events are **independent** if one occurring has no bearing on the probability of the other occurring.
    - Formally, $\mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\mathbb{P}(B)$.
- If A and B independent, then $\mathbb{P}(A \mid B) = \mathbb{P}(A)$
    - Knowing B occurred doesn't change the probability of A

Sampling and independence

- Sampling > 1 with replacement: **independent draws**
  - Randomly draw 1 senator, note the name, then put it back in hat.
  - Shuffle, randomly draw 2nd senator, note the senator.
  - First draw doesn't affect second $\rightsquigarrow$ independence
- Sampling > 1 without replacement: **dependent draws**
  - Randomly pick 1st senator, note name, leave it out.
  - Randomly pick 2nd senator from remaining 99 senators.
  - First draw affects the probabilities of the second.